Scientific Aspects of Juggling

Claude E. Shannon

> "Do you think juggling's a mere trick?" the little man asked, sounding wounded. "An amusement for the gapers? A means of picking up a crown or two at a provincial carnival? It is all those things, yes, but first it is a way of life, friend, a creed, a species of worship."
>
> "And a kind of poetry," said Carabella.
>
> Sleet nodded. "Yes, that too. And a mathematics. It teaches calmness, control, balance, a sense of the placement of things and the underlying structure of motion. There is a silent music to it. Above all there is discipline. Do I sound pretentious?"
>
> Robert Silverberg, *Lord Valentine's Castle.*

The little man Sleet in Silverberg's fantasy who so eloquently describes the many faces of juggling is a member of a juggling troupe on a very distant planet, many centuries in the future. We shall discuss some of the many dimensions of juggling on our tiny planet Earth from the viewpoints of Darwin (What is the origin of the species *jongleur?),* Newton (What are the equations of motion?), Faraday (How can it be measured?) and Edison (Can American inventiveness make things easier?). But we shall try not to forget the poetry, the comedy and the music of juggling for the Carabellas and Margaritas future and present. Does this sound pretentious?

On planet Earth, juggling started many centuries ago and in many different and distant civilizations. A mural on the Egyptian tomb of Beni-Hassan dating back to 1900 B.C. shows four women each juggling three balls (Fig. 1). Juggling also developed independently at very early times in India, the Orient, and in the Americas among the Indians and Aztec cultures.

The South Sea island of Tonga has a long history of juggling. George Forster, a scientist on one of Captain Cook's voyages, wrote:

> "This girl, lively and easy in all her actions, played with five gourds, of the size of small apples, perfectly globular. She threw them up into the air one after another continually, and never failed to catch them all with great dexterity, at least for a quarter of an hour."

The early Greek historian Xenophon, about 400 B.C., describes in *The Banquet* the following incident.

> "At that, the other girl began to accompany the dancer on the flute, and a boy at her elbow handed her up the hoops until he had given her twelve. She took these and as she danced kept throwing them whirling into the air, observing the proper height to throw them so as to catch them in regular rhythm. As Socrates looked on he remarked: 'This girl's feat, gentlemen, is only one of many proofs that woman's nature is really not a whit inferior to man's, except in its lack of judgment and physical strength.' "

Fig. 1. Egyptian wall painting, circa 2040 B.C. (Source: unknown.)

This excerpt is interesting at a number of different levels. At the juggler's level, did the girl in fact juggle twelve hoops at once? This is an astounding feat — in the twenty-three centuries since then no one has reached this record, the highest being the great Russian juggler, Sergei Ignatov, who does nine regularly and sometimes eleven in his act (Fig. 2). However, who could ask for better witnesses than the great philosopher Socrates and the famous historian Xenophon? Surely they could both count to twelve and were careful observers.

At a different level, it is amusing to note how Socrates, departing from his famous method of teaching by question, makes a definite statement and immediately suffers from foot-in-mouth disease. Had he but put his period nine words before the end he could have been the prescient prophet of the women's equality movement.

In medieval times court jesters and traveling minstrel troupes often combined the three arts of juggling, magic and comedy. A ''Street of the Conjurers,'' where daily performances could be seen, was a feature of many cities. Early in the present century vaudeville, circuses and burlesque were a spawning ground for jugglers. Many jugglers of this period combined a comedy patter with their juggling and some of the great comedians, Fred Allen (billed as the World's Worst Juggler), Jimmy Savo (I Juggle Everything from a Feather to a Piano!) and W.C. Fields (Distinguished Comedian & Greatest Juggler on Earth, Eccentric Tramp) evolved from this background.

Jugglers are surely among the most vulnerable of all entertainers. Musicians and actors can usually cover their slips but if a juggler makes a mistake ''it's a beaut!.'' This has led through the centuries to a vast number of comedy lines and cover-ups for the missed catch or the dropped club. Books on juggling contain whole sections about how to save face in such situations. La Dent introduced the ''Swearing Room,'' a boldly labeled, screened-off portion of the stage to which he would retreat on missing a trick. Others wear numerous medals which they take off if they make a slip. One writer suggests a large pair of spectacles to put on after an error. There are dozens of comedy lines to be used after an error, e.g., ''That's part of the act, folks — the part I didn't rehearse.'' W.C. Fields was a master of the seemingly wild throws, dropped balls and incredible recoveries. Expert jugglers could not distinguish Fields' intentional moves from actual misses.

This very vulnerability may have led to the dichotomy between the comedy and the technical jugglers. The technicians aim for perfection, striving to keep more and more objects in the air at one time, the ''Numbers game'' in juggling parlance. Albert Lucas, keeping score

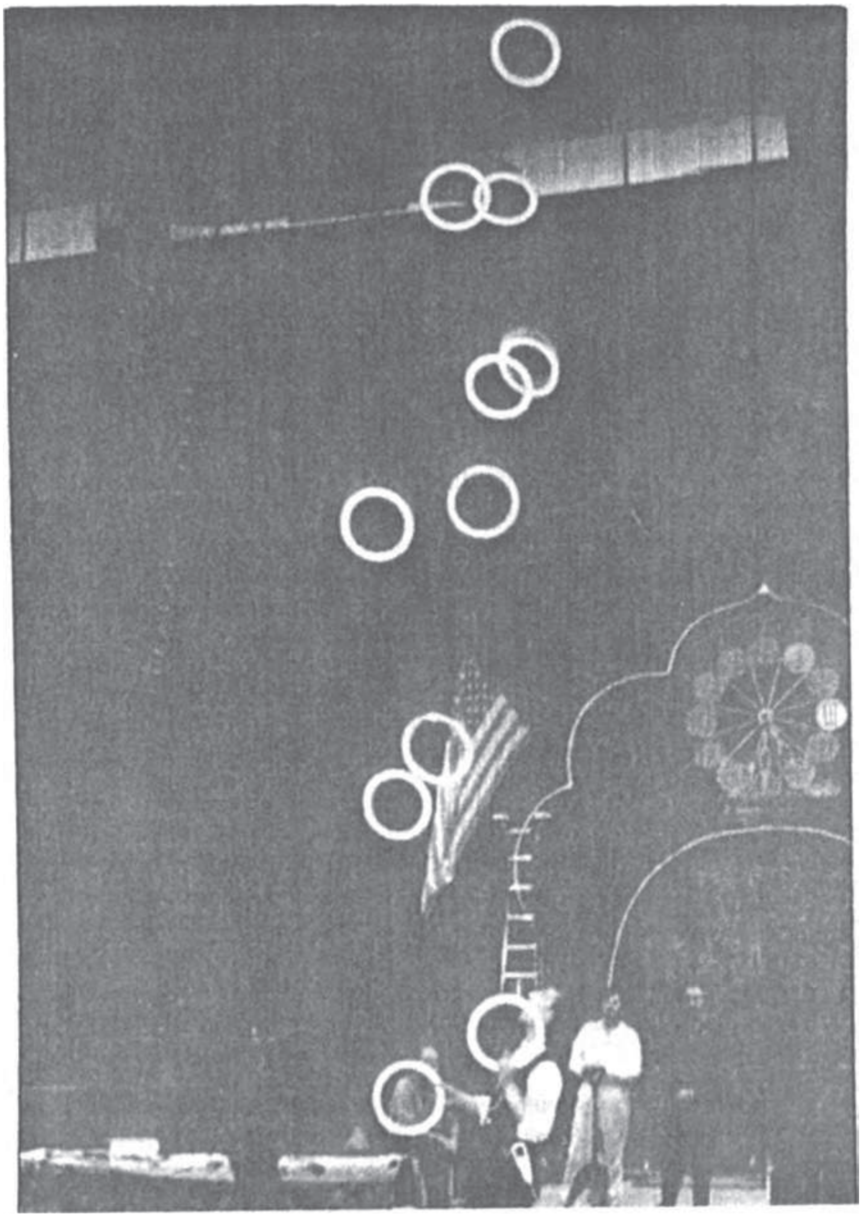

Fig. 2. Sergei Ignatov took American juggling audiences by storm when he toured this country with the Moscow Circus in 1978. Ignatov demonstrated his mastery of seven balls and five clubs by doing a number of variations with them during his performances. He performed with eleven rings for his show-stopping finale. (Source: Jugglers World Calendar, 1980. Photo courtesy Roger Dollarhide.)

of his own performance, reported "2 misses in 46,000 throws." Outstanding among performers in this area was Enrico Rastelli, who was said to have juggled ten balls. He could do a one-armed handstand while juggling three balls in the other hand and rotating a cylinder on his feet.

There have been many talented women jugglers. We have already mentioned the Grecian lady who was perhaps the numbers champion of all time. Some others are Lottie Brunn (Fig. 3), the only woman to perform solo in Ringling Brothers' center ring, and the child star Trixie from Germany. A photograph in the book *Will Mariner* by Somerville shows a woman on Tonga showering eight balls — an incredible feat (showering being a very difficult way of keeping many balls going).

With the advent of the various electronic media (radio, motion pictures and television), vaudeville and the circuses were doomed and juggling went into decline. This lasted several decades and many of the great jugglers sought greener pastures.

Recently, however, there has been a revival of this ancient skill, particularly among young people of college age. The University of Colorado, Pennsylvania State University, M.I.T., Wesleyan and many other universities have active juggling groups. The market for the professional juggler is still small — some ice shows, Las Vegas floor shows, the Ringling circus. Street performers may be seen in such places as Greenwich Village, Harvard Square and San Francisco. However, the main thrust of juggling today is that it is a challenging recreation for amateurs. It can involve not just one person trying to juggle five balls but two people passing six or more clubs between them, or, indeed, twenty people in a very complex pattern of club passing.

Juggling seems to appeal to mathematically inclined people — many amateur jugglers are computer programmers, majors in mathematics or physics, and the like. One knowledgeable juggler, a mathematics professor himself, told a reporter that forty per cent of jugglers are "algorithmically inclined." In the *New York Times* story, it appeared as "forty per cent are logarithmically inclined." (This curious anagram probably conveyed the meaning better than the original to most readers.) In spite of this appeal to the technical mind, there seems to be very little mathematical or scientific literature relating to juggling.

Juggling also appears to be a skill that very young people can master well. W.C. Fields was a fine juggler at age fourteen. Albert Lucas, one of the world's top performers, was brought up in a circus family and is said to have juggled five balls at age five. He is now nineteen and has performed several years with the Ice Capades (Fig. 4). (Note not only the nine rings but the mouthpiece and ball and the ring on one leg.) Other young performers are Demetrius and Maria Alcarese, who in their teens show excellent potential. In the various 1980 competitions of the International Jugglers' Association, the Juniors (less than three years' experience), Seniors, seven-ball, five-club and team events were all won by young people in their teens or early twenties. Especially impressive were Barrett Felker and Peter Davison who, with Keziah Tannenbaum, won the team event, and battled it out between themselves in five-club juggling. It seems plausible that juggling ability may have its child prodigies in much the same way as music (Mozart, Mendelssohn and Menuhin), mathematics (Pascal, Gauss, Abel) and chess (Morphy, Capablanca, Reshevsky, Fischer). Further, it seems likely that the age of peak ability for juggling may be quite young, as it is for such sports as gymnastics and swimming.

The Tools of Jugglers

It is well known that witches use three basic tools in their craft — the bell, the book and the candle. Jugglers, whom many think are close cousins of witches, also use three basic tools — the ball, the ring and the club.

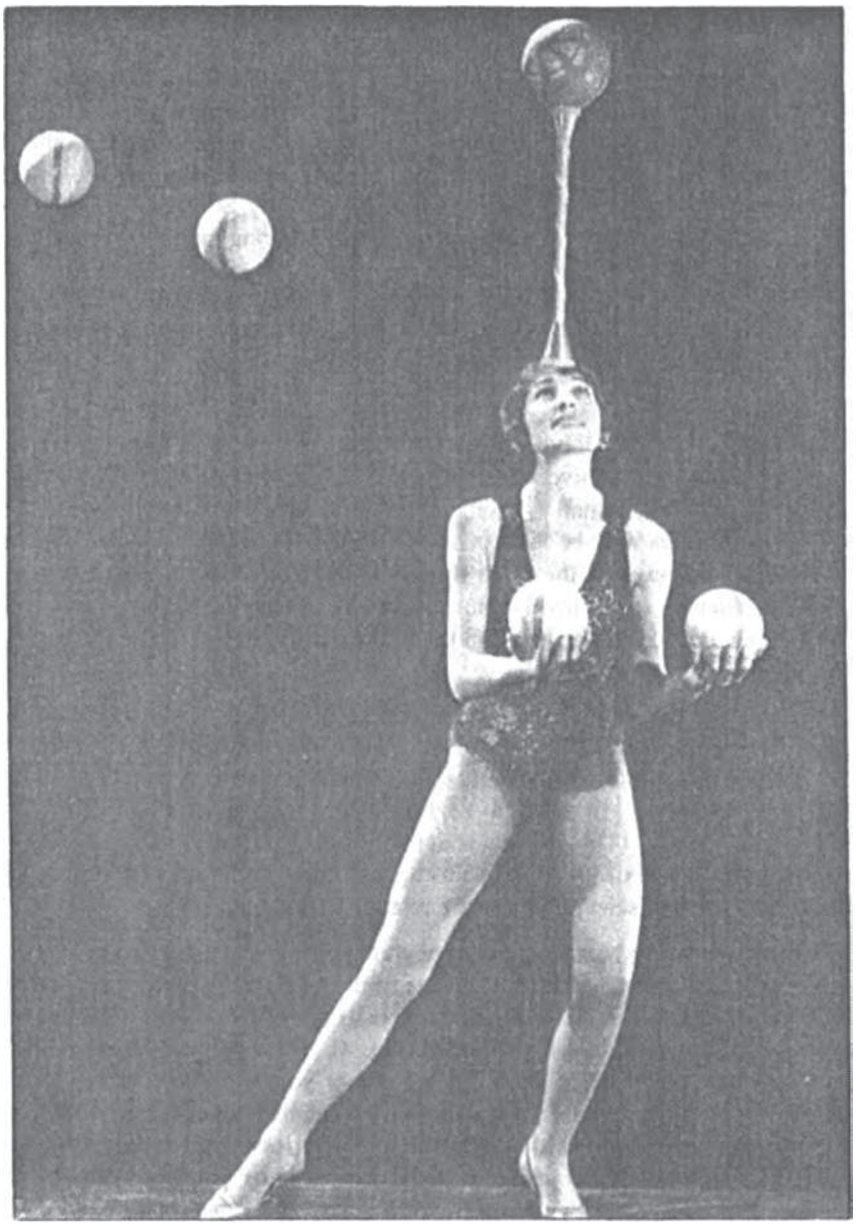

Fig. 3. Lottie Brunn was billed by Ringling Brothers as the "Greatest Girl Juggler of All Time." She is also the only female juggler to have performed solo in the center ring. She has appeared at all of the top Las Vegas nightclubs, as well as Radio City Music Hall in New York. (Source: Jugglers World Calendar, 1980. Photo courtesy Lottie Brunn.)

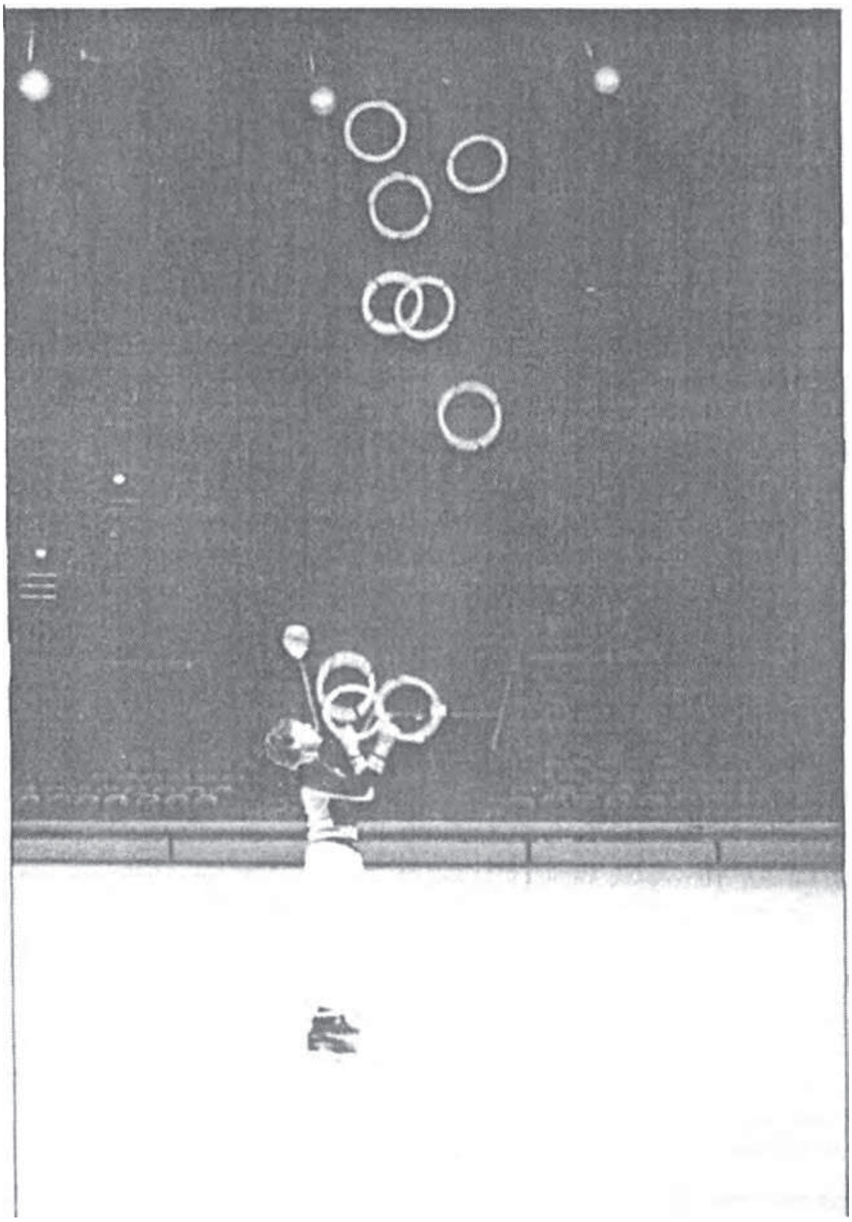

Fig. 4. Albert Lucas began juggling as a small child, and could juggle five balls by the age of five. (Source: Jugglers World Calendar, 1980. Photo courtesy Robert Leith.)

Balls are the easiest for learning, and lacrosse balls weighing about five ounces and two and one half inches in diameter are very popular among jugglers. However, all sizes of balls have been used, from jelly beans to basketballs. Recently a record was set in juggling three eleven-pound bowling balls for thirty-seven catches. At the other end of the weight scale, the Mexican juggler Picasso has juggled up to five ping-pong balls by popping them out of his mouth.

Most professional jugglers like rings. If the juggler stands at right angles to the audience, the rings present a large viewing area and are easily seen, as in Fig. 2. Furthermore, when tossed, the area presented to the juggler is less than half that of balls, causing less interference. This probably leads to higher numbers in ring juggling than in ball juggling.

While balls and rings are ancient, it appears that clubs date only to the late 19th century. At that time the art of swinging "Indian clubs" became popular. This is an elegant and demanding art form in itself. It was left to the jugglers, however, to discover the possibilities of tossing clubs, and, even more exciting, exchanging them with partners. The sight of two people with six rapidly spinning clubs being passed back and forth was electrifying. The early clubs were turned from solid wood and were quite heavy. Modern clubs are made of plastic, often with a wooden dowel through the middle, or of fiberglass. They weigh much less than the wooden ones, about nine ounces, and are about nineteen inches long.

We spoke earlier of the witches' basic tools, bell, book and candle. Their secondary instrumentation would have to include cauldron, broomstick and the entrails of a toad. If these seem a bit exotic, consider the objects that jugglers use.

Jugglers will juggle with almost anything — gossamer scarves, kitchen utensils, flying saucers, badminton racquets and flaming torches. One three-ball juggler uses a rubber chicken, a head of lettuce and an M&M candy. Probably the ultimate in choice of weapons is that of the Flying Karamazov Brothers, a totally insane group of performers, mad even by jugglers' standards. They announce that they are going to juggle with a chainsaw and bring out a completely harmless toy chain saw. However, they follow with a real machine operating at full speed and toss this around.

Flying saucers and gossamer scarves involve aerodynamics — the interaction of the object in flight with the air. The most common juggling objects, rings, balls and clubs, are sufficiently massive relative to their surface area, and their speeds sufficiently slow, that such effects can be ignored.

At the instant a juggling object leaves contact with the hand of the juggler it enters the world of analytical dynamics, free of the juggler's control but subject to the laws and theorems of Newton, Euler, Poinsot and Poincare. It seems unlikely that any of these mathematicians ever did a three-ball cascade, but their differential equations describe not only the complex motion of planets and satellites but the "polhodes," "herpholodes" and "invariable lines" of a juggler's club in motion.

What can we learn from mechanical principles about the motion of a club? First, and simplest, its center of gravity will follow a parabolic course. The relation between height and time will be given by $h = \frac{1}{2} g t^2$, where h is measured from the apex of the trajectory and $g = 32 \mathrm{ft/sec}^2$ is the acceleration of gravity. For example, if a juggler throws a ball two feet in the air and catches it with his hand at the same level, two feet below its highest point, the time of flight would be $2\sqrt{2h/g}$ or .7 seconds. The horizontal distance traveled is of no significance. It will take precisely the same time going straight up and down or to a juggling partner, say six feet away, as long as the vertical travel is the same.

If the object is tossed to a height h_1 and caught after a drop of h_2, the time of flight would be given by $\sqrt{2h_1/g}+\sqrt{2h_2/g}$. In most cases of juggling, h_1 and h_2 are close enough the approximate formula $2\sqrt{2\bar{h}/g}$ can be used, where $\bar{h}$ is the average of the two distances. The following table shows the time of flight F for various values of $\bar{h}$ ranging from six inches (a fast three-ball cascade) to the sixteen feet we estimate for Ignatov's eleven rings in Fig. 2.

$\bar{h}$	6″	1″	2′	4′	8′	16′
F (sec)	.35	.5	.71	1	1.41	2

Much more complex than the motion of the center of gravity is the rotational motion of an object in free flight. This is a subject that was studied extensively by many mathematicians in the 18th and 19th centuries. The motion will depend on the distribution of mass of the body, which can be summarized in its ellipsoid of inertia. It is noteworthy that the three favorite juggling objects are very special with regard to their ellipsoids of inertia. The ball has all three axes of inertia equal, the ring has two equal and the other very small, and the club has two equal and the third very large. These three special situations lead to much more predictable behavior than that of an object with three unequal axes. In the latter case, the object will show stability in rotation about the largest and smallest axes but not about the intermediate one. It is easy and interesting to observe this property. Put a rubber band around a book so it cannot fly open. Now toss it up, with a spin, in each of the three possible ways. The book will spin stably about its shortest dimension and its longest dimension, but when spun about the intermediate dimension will continue turning almost fully around in a most erratic fashion.

Basic Juggling Patterns

"The cross rhythm of 3 against 2 is one of the most seductive known." So wrote Gene Krupa, the great jazz drummer, some forty years ago. Seductive it is, whether it be the Chopin F minor Etude or Krupa's own tom-tom chorus in "Sing-Sing-Sing" with its ever-changing emphasis.

The visual analog of the three against two rhythm is in the juggler's three balls into two hands, the three-ball cascade. This is the first pattern that most people learn and the most fundamental, and it is capable of as many changes as bell ringing.

Fig. 5a shows how the simplest three-ball cascade appears to a viewer. Jugglers can vary the height of the throw, the width of the hands and even reverse the direction of motion. They can make under-leg throws, floor bounces, behind-the-back throws, overhand (claw) catches, "chops," and numerous other variations.

The three against two can be done with clubs, rings, and, in fact, almost anything. The Brothers Karamazov in their act offer the audience a choice of three from a wildly varied assortment of possibilities — such things as hoses, basketballs, clubs, candies, etc. The audience by their applause picks three of these and the "world's greatest juggler" attempts to keep them in the air. Should he fail, he gets a pie in the face.

Many expert jugglers restrict their acts to three-ball variations and entire books have been written on this subject. At the 1980 jugglers' convention there were seventeen entries in the Seniors competition — each entrant was allowed six minutes for an unrestricted act, judged by seven experts and viewed by a sophisticated audience of several hundred of his peers. The acts were all good and ranged from comedy monologues to highly technical exhibitions.

The first prize, however, went to a young man, Michael Kass, who used no props other than

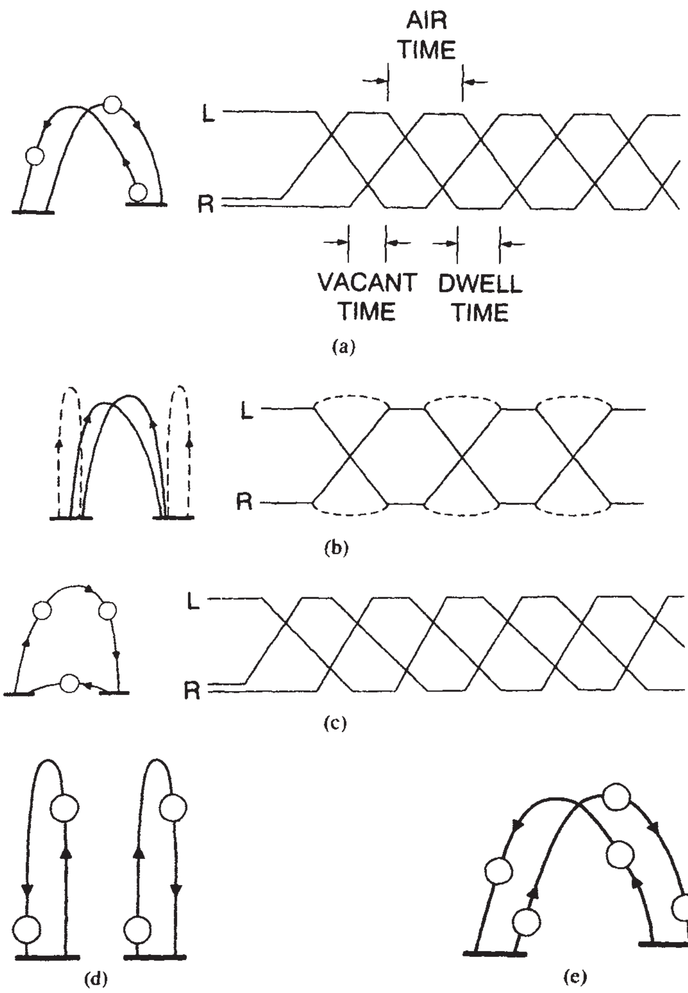

Fig. 5. (a) Three-ball cascade. (b) Two balls. (c) Three-ball shower. (d) Three-ball fountain. (e) Five-ball cascade.

the three clubs he was juggling as he came on the stage. His routine consisted almost entirely of variations on the three-ball cascade. The clubs were double-spun, thrown behind his back, under his legs, in chops, flats and other variations to the music of Synergy "Synthesizer." His stunning climax came with a series of "foot-drops" where the clubs were dropped to his feet and tossed back into the cascade with a flip of the ankle. The audience was hypnotized and gave him a standing ovation.

While the three-ball variations can be totally fascinating, the juggling repertoire has evolved a vast series of other patterns. A small sampling of these is shown in Fig. 5. The right-hand parts of these diagrams show how the juggled objects progress from hand to hand with time.

The almost trivial example of two balls and two hands (Fig. 5b) is included because of its theoretical importance. It is the simplest case where a choice can be made at each toss — whether to interchange the balls or keep them in the same hands.

The three-ball shower (Fig. 5c) is similar to the three-ball cascade but with different timing — the left to right throw going almost horizontally so that the whole pattern looks more like balls rotating in a circle.

Four balls are usually done in one of two ways: the "fountain" (Fig. 5d) where the two hands are out of synchronism and which, in fact, amounts to two balls in each of two hands which never interchange, or a synchronous movement where the balls can be interchanged or not at each toss.

Fig. 5e shows the pattern for a normal five-ball cascade, a natural generalization of the three-ball. In a multiplex juggle two or more balls may be caught in one hand at the same time. In the 1979 competitions, one of the contestants raised a considerable furor by juggling seven balls with the multiplex system. Multiplex juggling in itself is an interesting and picturesque art but considerably easier than the standard form where only one ball contacts a hand at one time. After some soul-searching, the IJA decided to disallow multiplex juggling in their numbers competitions.

The Uniform Juggle

We shall define a *uniform juggle* as one without multiplexing and with all dwell times the same (D), all flight times the same (F) and all vacant times the same (V). Uniform juggles include many of the most common juggling patterns, the three-, five- and seven-ball cascades, two or three in one hand, the four-ball fountain and many passing routines among two or more jugglers. At a recent juggling convention with perhaps one hundred jugglers on the floor it appeared that 75% or more were working on uniform juggles.

Uniform juggles have good reason for their popularity — all of the hands do about the same thing: they throw balls to the same height, hold them for the same time and are vacant for the same time.

The same uniform juggle, for example the three-ball cascade, may of course appear in a multitude of forms. The juggler may do overhead (clutch) juggling, he may toss the balls under his legs or behind his back or cross his arms in a bewildering fashion. For our purposes we are concerned only with the uniformity of the time parameters D, V and F.

Focusing on the uniform juggle is somewhat akin to the geometer who spends much time with circles and triangles. He is well aware of the existence of other geometric figures, but the simple structure of these leads to elegant mathematical theorems. We shall later give some results relating to generalization to other types of juggles.

Of course a juggle may be uniform only for a period of time. In fact, many juggling routines are made up of segments, which are uniform in our sense, with transitional moves. *The theorems which follow about uniform juggles require only that uniformity last for the period that it would take one ball to visit all the hands* (if it did them all in sequence, that is, $H(D+F)$). These theorems will be first stated and discussed, and later proved in a general argument.

Theorem 1. In a uniform juggle with dwell time D, vacant time V, flight time F, then

$$\frac{F+D}{V+D} = \frac{B}{H},$$

where B is the number of balls and H the number of hands.

In one sense this theorem seems almost trivial. It states a proportionality between the number of balls and hands, and the total time for each ball circuit ($F+D$) and each hand circuit. It is, however, more subtle than it appears, its proof depending on the uniformity of the juggle for at least H tosses.

Theorem 1 allows one to calculate the range of possible periods (time between hand throws) for a given type of uniform juggle and a given time of flight. A juggler can change this period, while keeping the height of his throws fixed, by increasing dwell time (to increase the period) or reducing dwell time to reduce the period. The total mathematical range available for a given flight time can be obtained by setting $D = 0$ for minimum range and $V = 0$ for maximum range in Theorem 1. The ratio of these two extremes is independent of the flight time and dependent only on the number of balls and hands.

Corollary. In a uniform juggle with a fixed flight time, the range of possible periods is $B/(B-H)$.

For example, in the three-ball cascade a 3-to-1 period (or frequency) ratio is possible; with five balls the maximum ratio is 5-to-3. With larger numbers of balls there is much less range available. At nine, it is 9-to-7. Of course, in actual juggling the possible range will be considerably smaller than these figures — it is not possible to have either zero dwell time or zero vacant time.

Theorem 2. If B and H are relatively prime (have no common divisor) then there is essentially (i.e., apart from labeling) a unique uniform juggle. The balls can be numbered from 0 to $B-1$ and the hands from 0 to $H-1$ in such a way that each ball progresses through the hands in cyclical sequence and each hand catches the balls in cyclical sequence.

Theorem 3. If B and H are not relatively prime, let n be their greatest common divisor with $B = np$ and $H = nq$ (p and q relatively prime). Then there are as many types of juggles as ways of partitioning n into a sum of integers.

For example, if n were 5, a partition into $2+2+1$ would correspond to three disjoint juggles. There would be no possible interchange of balls among these three juggles. Each "2" in this partition would correspond to $2p$ balls circulating among $2q$ hands in a fashion similar to the cases described above, except that at each toss there is a choice of two possibilities. The "1" in this partition would be a cyclical juggle of the type in Theorem 2, with p balls circulating around q hands with no choice.

In the common case of two jugglers, each with three balls (or clubs), we have $B = 6$ and $H = 4$. The greatest common divisor is 2, which can be written as a sum of positive integers in two ways: 2 or $1+1$. The case of 2 corresponds to the jugglers starting simultaneously. Thus at each toss there is a choice of two possibilities: a self-throw or a throw to a partner. In group juggling, incidentally, a downbeat analogous to that of a musical conductor is used to ensure synchronism.

The other partition, $1+1$, corresponds to two jugglers out of synchronism. There is no way to pass clubs from one pair of hands to the other without violating the uniform juggle condition.

The number of partitions of n into a sum increases rapidly with n as the following table shows:

n	1	2	3	4	5	6	7	8	9	10
no. of partitions	1	2	3	5	7	11	15	22	30	42

We now prove these theorems.

Suppose that at time 0 a uniform juggle commences with the toss of a ball. Let us follow this ball for a period of H catches (a time $H(F+D)$). Since there are only H hands and the ball has visited $H+1$ (counting the one it started with) it must have visited the same hand twice. This is sometimes called the pigeonhole principle in mathematics — if you have $n+1$ letters in n pigeonholes, there must be some pigeonhole with at least two letters. The pigeonhole principle, or *Schubfachprinzip*, as German writers call it, applies only for finite numbers. The theory of transfinite juggling, like the case of the Giant Rat of Sumatra, is a story for which the world is not yet prepared.

Focusing now on the hand that was visited twice by a ball, suppose the number of catches of the ball was a. Then the time between these catches was $a(D+F)$. Meanwhile, the hand, to satisfy the uniform juggling condition, has made some integer number, say b, catches in the same time, and consequently $b(D+V) = a(D+F)$, or $(D+F)/(D+V) = b/a$, a rational number. In other words, the times related to balls, $F+D$, must line up exactly with the times related to hands, $D+V$, after some integer number of throws.

Let a/b be reduced to its lowest terms, say p/q. Thus p and q would be the smallest number of hand catches and flight times which could return to coincidence.

Now consider the set of all balls, say d_1, thrown at the time t_0. These may be called a synchronous set — with a uniform juggle they will all be caught at the same times and thrown at the same times so long as the juggle remains uniform. Furthermore, no other balls can fall into this time pattern under the uniformity condition. Consider now the subset of balls caught by the hands that caught this synchronous group at a time interval $(D+V)$ after time t_0, that is, the next catches of these hands. These balls also form a synchronous set at a later time. The same may be said for each hand-catching interval $(D+V)$ until we reach $p(D+V)$ from time t_0, the first time it is possible for the original balls to return to the original hands. At this time, all of the original balls *must* have returned to the original hands, since no other balls could have come into synchronism with these and all of these are timed right for these hands.

Consider now the balls thrown by the subset of hands we have been discussing at the time $(D+V)$ after t_0 (one throw after our initial time). This subset must basically parallel the subset we have just described — these balls go to the same next hands and after p throws return as a subset to the same original hands.

Continuing this argument, we obtain pd_1 balls following each other cyclically through p stages, d_1 balls at a time.

If d_1 did not exhaust all of the balls, we carry out the same process with the next balls caught after time t_0. These are, in a sense, totally disjoint from the first subset. They cannot be caught by the hands involved in the first sequence. Consequently we obtain a totally disjoint subset of the balls juggled by a totally disjoint subset of the hands.

Continuing in this fashion, we exhaust all of the balls and all of the hands. The total number of balls, B, is therefore equal to $\sum pd_i$, and the total number of hands, H, $= \sum qd_i$.

The Jugglometer

"Some son-of-a-bitch will invent a machine to measure Spring with"

e. e. cummings

In order to measure the various dwell times, vacant times and flight times involved in actual juggling, some instrumentation was developed. The first of these devices used three electromagnetic stopclocks, accurate to about .01 seconds (10 milliseconds). These were activated by a relay circuit. The relays were connected to a flexible copper mesh which was fitted over the first and third fingers of the juggler's hand. Lacrosse balls were covered with conducting foil. When one of these was caught it closed the connection between the two fingers, causing a clock to start. Break contacts allowed the measurements of vacant times, and contacts on two hands enabled measurement of flight times. Juggling rings and clubs were also made conductive with foil so that the various times could be measured.

While this system was workable, it required several people to observe and record the observations, and we replaced it with a computerized version using an Apple II computer. This uses essentially the same sensor arrangement, copper sleeves for the fingers connected to the "paddles" of the computer. A computer program displays the time in milliseconds of the various tosses and catches.

Preliminary results from testing a few jugglers indicate that, with ball juggling, vacant time is normally less than dwell time, V ranging in our measurements from fifty to seventy per cent of D. Of course, jugglers have great freedom in changing this ratio, especially when juggling a small number of balls.

With a three-in-one-hand juggle, a dwell of .27 sec. against a .17 sec vacant time was measured. With three clubs, typical figures were .52 sec dwell, .20 sec vacant and .56 sec flight time. Clubs lead to a larger ratio of dwell to vacant time because of the need to stop a club's rotation and start it again. At each catch the spin energy is all dissipated and must be supplied anew at each toss. Curiously, after all of this the club is spinning in the same direction relative to the juggler.

How Much Do Jugglers Weigh?

Watching a competition of jugglers working with eleven-pound bowling balls recently reminded me of an old puzzle which I shall restate as follows.

Claude Crumley comes upon a canyon. He is carrying three copper clappers from his latest caper. Claude weighs 98 kilo and each clapper 1 kilo. The bridge across the canyon can carry 100 kilo. How can Claude cross the canyon?

The intended answer is that Claude walks across doing a three-clapper cascade. I surely hope he doesn't try this, for he would be catapulted into the canyon. All of the gravitational forces on a juggler's objects in the air must be supported basically through his feet, by way of the larger forces downward when he accelerates the objects upward. Put another way, the center of gravity of the entire system, juggler plus objects, would accelerate downward were not an average force of the weight of this system applied upward, and this can only come via his feet. In Fig. 2, Ignatov's feet are pressing down with an average force of his weight plus that of the eleven rings, just as surely as if he had the rings around his neck, and just as surely as Isaac Newton sat under that apple tree.

Why Does Juggling Get So Hard So Fast?

Most people can learn to make 20 or 30 catches of a three-ball cascade within a week or two, and some learn within a few hours. Moving up to four balls (basically two in each hand) again is not extremely difficult. People do this in a few weeks of practice. The five-ball cascade is a different matter indeed. I have asked many five-ball jugglers how long it took them to learn. The answers ranged from six months to two years, and these were talented and dedicated jugglers. Six balls, three in each hand, is again a step forward in difficulty, and seven balls is a point that very few jugglers reach. At the 1980 Fargo convention, a competition was held for juggling seven objects (balls, rings or clubs). Only six entered, and the longest time registered was 5.6 seconds, twenty-six catches. A number of other seven-object jugglers are known who were not entered or present at this convention but the number is very small, probably less than ten. We estimate that fewer than twenty-five people in the United States can juggle seven objects for more than twenty-five catches.

Moving into the stratosphere, we come to the superstars of juggling. Among these is the legendary figure Enrico Rastelli, who was famous for many complex balancing acts and is said to have juggled ten balls. He practiced his craft as musicians do theirs, ten hours a day, and died at only 35. The current world champion in the "numbers game" is undoubtedly the great Soviet star of the Moscow circus, Sergei Ignatov. He juggles nine rings very securely in his act, and can do eleven rings on occasion. Another strong contender is the American, Albert Lucas, who started very young in a circus family and now juggles ten rings. In the picture, he has nine rings but note also the ring rotating on his leg, the ball balanced in his mouth and that he is balanced on one ice skate, truly an incredible juggling feat.

Why does it get so hard so fast?

To begin with, suppose a juggler requires the same dwell time and vacant time whether he is juggling three balls or nine. From Theorem 1, thinking of all terms fixed except the number of balls and flight time, we see that the flight time goes up linearly with the number of balls. Now flight time, as we have seen, increases only as the square root of height of throw, which is proportional to energy. Thus already we are facing energy and height requirements going up roughly as the square of the number of objects juggled.

However, the situation is much worse than this. There is much painful positive feedback at work. First, throwing objects higher will require longer dwell time to accelerate. Second, there is always dispersion in angles of toss. With the same dispersion of angle, the dispersion of where the objects land increases in proportion to the height of throw. The juggler, therefore, will have increasing difficulty in catching, and consume more time. Even more serious is the dispersion in vertical velocity of the toss. This can cause two objects, one thrown a little high and the next a little low, to come down at almost the same time, making it impossible to catch them both.

All of these factors must react on each other — the dispersion of angle and flight time forces greater dwell and vacant time, which in turn requires higher throws. In the higher numbers this vicious loop can only be controlled by the most precise throwing in height, in angle and in tempo.

Bounce Juggling

> "— things never fall *upwards*, you know. It's a plan of my own invention. You may try it if you like."
>
> Lewis Carroll, *Through the Looking Glass*

Bounce juggling is an interesting variety of juggling where the balls are thrown downward and bounced off the floor rather than tossed upward. It is possible to do all of the basic juggling patterns — three- and five-ball cascades and the like — in this upside-down fashion.

There are pluses and minuses to bounce juggling vis-a-vis toss juggling. First, in bounce juggling much of the energy expended on each throw is conserved. Balls of highly compressed rubber ("superballs") will rise to .85 of the original height. This means that the bounce juggler must supply only 15% of the energy that the toss juggler would for a given height of throw. It also probably implies less dispersion in both time and direction, since these tend to be proportional to energy requirements.

On the negative side, in bounce juggling the juggler's own hands interfere with his line of sight to the juggled balls. In addition, the part of the trajectory of the balls which we think most

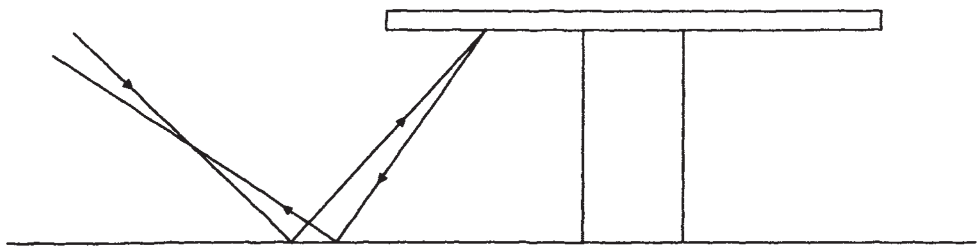

Fig. 6. "Superball" bouncing under tabletop.

optimal for prediction, namely halfway between the toss and the catch point, is now on the floor at a very poor viewing angle. When tossing, the eyes are in a fine position for viewing the tops of the parabolic orbits.

All told, it seems likely that bounce juggling may be easier than toss juggling, but very few people specialize in it. I have seen bounce jugglers work five balls at less than two feet from their bouncing surface — faster than these same jugglers could do a five-ball toss cascade.

There are other intriguing possibilities in bounce juggling. If the juggling ball has a high coefficient of friction with the floor (as do "superballs"), when it is thrown at an angle against the floor a considerable fraction of the horizontal component of energy is turned into rotational energy. If it continues and strikes the floor again, the second bounce appears to be unexpectedly flat. More interesting, however, is to let it hit the underside of a table, Fig. 6. The rotational energy then causes it to very closely return on its original path, bounce again on the floor and return very closely to the hand that threw it. It is possible to bounce juggle several balls this way under a tabletop. Since each one makes three loud clicks on each throw, the audible effect approaches that of a drumroll.

In other words, this situation is a kind of boomerang where one can throw the ball quite arbitrarily and it will come back to the same place except for a slight drop. This is something of a juggler's delight, except that each ball returns to the hand that threw it. To get to the other hand, as in a normal cascade, requires moving the other hand to where the throwing hand was.

A generalization of this automatic return in bounce juggling which, however, gives interchange of hands can be constructed as follows.

If a light ray starts from one focus of an ellipse it will reflect back to the other focus. If that ellipse is rotated about its major axis we have a prolate spheroid, and the light ray from one focus in any direction in three-dimensional space will be reflected back to the other focus.

Although balls do not reflect exactly like light rays, their behavior is fairly close. Imagine a reflecting dish about 20″ long by about 18″ wide shaped as a portion of a prolate spheroid. The two foci would be about 4″ above the surface where the juggler's hands normally are. He can throw a ball anywhere with one hand so long as it hits the dish, and it will automatically come back to the other hand!