The Physical Demands of Numbers Juggling

Jack Boyce

Contents

Introduction

Whenever I practice in public I get a wide variety of people who notice and comment upon the tricks that I do. I'm sure many jugglers hear the same things: "Can you do 20?" or "I had a friend who did 9 chainsaws." It seems that people have a hard time gauging skills (or accurately recalling their friends' abilities) at which they have no direct experience. If I had never run before, I probably wouldn't guess that running a 4:00 mile is vastly more difficult than a 5:00 mile, and that a 3:00 mile is essentially impossible. The limits of human performance are not always easy to guess, and juggling is no exception.

How and why is numbers juggling so hard? The intention of this paper is to try to find some answers to the question. The first half is a description of what I consider the important phases of learning, and some suggestions for progressing through them. The second half is a technical discussion of the difficulties that numbers juggling poses during each of the phases, with the secondary goal of partially justifying the phases of learning as presented here. It is my hope that a more systematic study of numbers juggling will help us refine our practice techniques at various points along the learning curve, particularly those of us without the benefit of regular coaching.

Phases of Learning, or "Why Do I Drop?"

Numbers juggling places a variety of demands on the human body. Nonjugglers commonly believe that reaction speed, or sheer mental confusion, limits jugglers in their quest for more objects. However, in this respect the human mind exceeds expectations, given enough training. In my own juggling experience I feel mentally calm when the pattern has good form; I'm not consciously aware of either the individual throws and catches, or even my own corrections to the pattern. In reality, it is the inability to consistently make accurate throws that limits jugglers.

I simplify juggling by breaking it down into seven phases of learning. The idea is that juggling is a combination of many skills, and a juggler should focus on the "weak link" which is most limiting him or her. I describe them as "phases" since these skills place ever-greater demands on the human body and hence are typically mastered in the order given below. However, *it is to be understood that an activity as complex as juggling cannot be so neatly ordered and delineated as this list would imply.*

1. **Release-Limited:** Learning to hold the objects in one's hands and release them consistently.
2. **Strength-Limited:** Learning to launch the objects high enough for the pattern, while maintaining consistency on the release
3. **Timing-Limited:** Learning to throw to an even height so that the balls land in a regular rhythm.
4. **Reach-Limited:** Learning to throw so that the objects consistently land close enough to the desired catching position.
5. **Form-Limited:** Learning the proper form and "feel" of the pattern.
6. **Collision-Limited:** Learning to throw so that the objects do not collide in the air.
7. **Endurance-Limited:** The juggler can usually continue until tired arms cause a mistake.

(*Note:* I use the term "balls" above but believe that similar considerations will apply to other props as well, perhaps with additions. Clubs, for instance, require one to apply a consistent and accurate spin while throwing.)

An understanding of where you fit on this list with a given pattern can be determined by introspection, a friend/coach watching you, or a video camera. In any case the central question is, "why do I usually drop?" Are balls coming down so out of time that I can't compensate, or are they colliding in the air, or is the pattern asymmetrical or otherwise out of form, or do I simply drop because I'm tired? It can be difficult to assess things accurately while juggling, and it can be equally difficult to judge oneself objectively -- try getting a (juggler) friend to help by watching you for a while. This is obviously not a determination that can be made on the basis of a few, or even ten, attempts.

The Release-Limited Phase

The first goal when learning a new number is to figure out how to hold the objects so that you can release them consistently -- a pretty obvious point. From a physical standpoint it is the first few throws which are the most difficult to make, for a few reasons:

1. The first objects thrown are usually the most precariously-positioned in the hands, and are also the least familiar. If you are practicing 7 then after the first few throws it's just like releasing 5, which is presumably automatic.
2. The first throws are made with all the weight of the other props in your hands at the same time. In fact, on the first throw an arm has to work as hard as it would if it were throwing *all* of the balls in that hand to the same height.

Taken together these may explain why some jugglers throw the first few throws higher than normal. Slowing down the start gives time to concentrate on the throws as well as muster the required strength. Other tips: look at how other jugglers do things, and strive for simplicity and consistency in arranging the props in your hands.

The Strength-Limited Phase

As objects are added to the pattern it increases in height, roughly in proportion to the square of the number juggled. The power required scales in the same way, and at some point the human body reaches a strength-imposed limit (particularly with regard to the first few throws). You are in the strength-limited phase of learning when you can't "empty your hands fast enough". Therefore even a perfectly accurate juggler cannot juggle an arbitrarily large number.

Strength also plays a secondary role in determining throwing accuracy, as the human body throws in a more erratic manner when it is close to its strength limit. For example, say that a given pattern requires 90% of your body's power to launch. If you increase your strength to the point that now only 65% of your body's throwing power is required, then you are bound to throw more accurately. In like manner, many jugglers can lift-bounce 7 balls much longer than they can toss-juggle it. Although there is no fundamental difference in the accuracy or speed required for these two patterns, the less strenuous one allows you to consistently throw more accurately and maintain the pattern longer. Finally, increasing one's strength gives rise to greater hand speed, for those heroic saves.

For the purposes of strength training, the biceps are the most obvious target. It is generally a good idea to train the opposing muscle group (the tricep) as well in order to avoid injury. Other muscles to consider are the muscles which move the upper arm: the deltoids, pectorals, and lats, among others. I would guess these are less important however, since the upper arm moves very little while juggling (the one exception being ring juggling). If weightlifting is no fun, a substitute is to use wristweights on your lower arms as you juggle. I would recommend the sort with adjustable weight. Some of the same effect can also be achieved by juggling heavy balls, although this type of training may cause wrist injury in some people. The advantage of weightlifting is that it isolates a given muscle and works it harder, resulting in faster strength gains. Wristweights, while not as effective at increasing strength, work all the relevant muscles and have a few added benefits: (1) you can practice juggling at the same time, and (2) the wristweights have a tendency to eliminate extraneous motion in your arms as you juggle and also force you to throw very accurately. It's a toss-up.

The Timing-Limited Phase

Once you can release the balls consistently and get them to go high enough, the next two phases involve getting the throws good enough that they can be consistently caught and returned to the throwing point. There are two types of throwing mistakes to be controlled here: errors in the landing time of a ball, and errors in the landing position of a ball. I have broken these apart into separate learning phases since in my own experience timing is typically learned first.

When a ball lands at the wrong time it forces the juggler to use more hand speed than she normally does. For example, a ball landing late has less time than normal to get back to the throwing point, requiring extra hand speed. There are two causes of deviant landing times: (1) a ball was thrown at the wrong time, or (2) it was thrown too high or too low. Generally the latter mistake is the more relevant one, although it is important to look at one's throwing rhythm carefully, especially during the release.

Considering now only mistakes in throwing height, note the following point: with 3 cascade and 4 fountain a ball which is thrown too high is easily corrected -- just by waiting. In the case of 4 this will mess up the alternating rhythm between the hands, but this is not essential to the pattern and can be corrected at one's leisure. However, with 5 or more this simple means of correcting timing mistakes is not available, one of the reasons five feels qualitatively different from 3 and 4 (and is hard to learn). In effect, the presence of these extra balls coming down forces you to adhere to a relatively strict timing, in the short term. There is some maneuverability; if a ball is landing too late to return to the throwing point in time, for example, you can throw it somewhat later and lower than you ordinarily would and it will land fine the next time. The price you pay is that by delaying the throw you have less time to make the next catch -- again more hand speed. There is no substitute for a good throw.

How accurate do the throw heights need to be? The following table serves as a rough guide for the amount of variation that is tolerable in the heights of throws. It was calculated using the following criterion: if one ball is too high by the stated amount and the next ball from the same hand is too low by the stated amount, then the difference between their landing times is half what it should be. In other words, in this extreme case the juggler has only half the time to do a catch-throw-catch than she should, and if her throw heights are always within the listed value of the average then the timing will never get any worse than that. (Pattern heights for up to 9 balls are from measurements of jugglers and are extrapolated for 10 and higher.)

Number of Balls	**Pattern Height (m)**	**Allowed Height Error (cm)**
5	1.0	13
6	1.5	15
7	1.6	13
8	2.2	16
9	2.2	14
10	3.5	19
11	3.5	18
12	5.0	23

One way to work on timing with balls is the "drop test". Throw them as you would to juggle, but don't catch -- let them fall to the ground. Listen to the rhythm of the landing balls: is it even? Of course it will never be exactly even, but if you do this several times you will get an idea of your *consistent* timing errors. You may ask why not just catch the balls and save yourself the trouble of all that bending over. Beyond the fact that you could use the exercise, the problem there is that your mind is busy making the catches, not listening to the rhythm.

At this stage don't throw to "avoid" the objects in the air -- throw to the correct points in space. If it is difficult for you to ignore the other objects in the air, try closing your eyes to do the drop test, and be as even and consistent as possible in your launches. For more information, read Boppo's good description of the drop test in his Numbers Notes.

The Reach-Limited Phase

As described above, this is the stage of development where you are releasing well, the pattern is high enough, and the catching rhythm is ok, but the landing positions are too erratic and inaccurate to catch and pull back into reasonable throws. Most jugglers will have trouble if the balls don't land within 12 inches (30 cm) or so of the desired spot, although more spread can be tolerated if one is collecting the balls without throwing (as in a flash). A small target indeed when throwing high.

In fact, it works like this: imagine a circle with radius 12 inches (or whatever you think you need), centered on the catching position and oriented horizontally. Mentally lift this circle up by an amount equal to *four times* the height of the pattern. Now if you turned off gravity, this would be your target, both in terms of correct throwing angle and allowed angular error. This factor of four is due to the excess hang time a ball has at the top of its parabolic flight path.

If we assume a "catching radius" of 12 inches as well as the following pattern heights, we can use this construction to calculate the tolerable angular error. Bear in mind that this is an absolute maximum; if one is to have a reasonable probability of getting 40 or so throws in a row to hit the target then one's *average* angular error must be roughly a factor of 2 smaller. As a comparison, a throwing accuracy of about 1.4° is required to score a free throw in basketball without hitting the rim (nothing but net).

Number of Balls	Pattern Height (m)	Max. Allowed Error (degrees)
5	1.0	4.4
6	1.5	2.9
7	1.6	2.7
8	2.2	2.0
9	2.2	2.0
10	3.5	1.2
11	3.5	1.2
12	5.0	0.87

This phase of learning is where the rubber meets the road for high numbers. I would guess that only a very small handful of jugglers are beyond this phase with 9, and that nobody is with 10 or more. How does a juggler work on throwing accuracy? There is no magic bullet, but the following may be helpful:

1. Most importantly, remember that you aren't trying to make the objects avoid one another at this stage. Attempting to do so confuses the issue and is more likely to hurt than help. The basic fact is, if you're at this stage then you don't have the throwing accuracy to make the objects avoid one another in a meaningful way.

2. Visualize a hoop in space at the crossing point, and try to throw through this. Alternatively, you can try concentrating on the apex points. (There is no general consensus as to whether the crossing point or apex points deserve more attention.) Consult Boppo's Numbers Notes for lots of other ideas.
3. Clean living, lots of rest, and some type of pact with the devil involving first-born children and/or your eternal soul.

The Form-Limited Phase

Once the objects can be thrown accurately enough to catch regularly, it may be that the primary problem with the pattern is its form. Note that I say the *pattern's* form, not the juggler's, which may of course be examined at all points during the learning process. In fact there is no broad agreement as to the relative importance of the juggler's form: some great jugglers such as Sergei Ignatov swear by it, and some such as Anthony Gatto apparently ignore the matter with equally impressive results (when he juggles 7 clubs his left shoulder is roughly an inch higher than his right, for example). This is a subtle issue and is not resolved here.

However, it is important to realize that the pattern (especially the cascade) *does* have an ideal form, given one's choice of pattern height and width. Departing from this ideal form makes collisions more likely. The goal at this phase of the learning process is to first identify the correct form, and then work on adopting it in one's juggling.

An understanding of accurate juggling form can come from several basically equivalent sources. You can watch a good juggler on videotape, use computer animation software such as Jack Kalvan's Optimal Juggler, or consult my results below (which are not in graphical form). I have found empirically that the great jugglers always adopt a nearly ideal pattern form, presumably the outcome of a trial and error process during training. By emulating either them or a computer animation one can short-circuit a lot of this work -- why reinvent the wheel? For the cascade the important trend is that as the number of balls increases from 5 to 7 to 9 and beyond, the ideal throwing point moves away from the center, closer to the catching point.

One obvious way to learn the "feel" of correct form is to videotape yourself to get feedback. This can have the added benefit of documenting it when you accomplish something good. Some jugglers have adopted the practice of recording themselves all the time. When reviewing tape, pay particular attention to the positions of the throw points.

I have noticed with myself that sometimes in the process of avoiding collisions my throw points wander away from the ideal, which of course just makes collisions that much more likely. Sometimes we benefit from ignoring the balls in the air a little (Jason Garfield says, "let the pattern come to you", which I interpret to mean: don't let inaccurate throws pull you around too much). Proper form can be a frustrating thing to achieve -- on some days my form is terrible unless I really concentrate, on other days it happens naturally. I still have no idea what makes a good day and what makes a bad one, but I have noticed that tiredness can lead to bad form.

Also according to some theories of the mind, a lot (perhaps most) of your learning occurs when you aren't actually practicing. If you believe this then it's a good idea to finish practicing a trick on a solid-feeling run, as this will be freshest in your mind. You can also try visualization exercises, or watching a juggling video.

The Collision-Limited Phase

There are two ways to avoid collisions in a pattern: passively and actively.

By the first method I mean simply that if your throws are accurate enough the balls will naturally miss one another, with no particular effort at "collision avoidance" on your part. In a later section I will quantify the accuracy required to achieve this. For now the important results are summarized by three points:

1. My results regarding collisions do not apply to rings.
2. Above I showed a table indicating the throwing accuracy required to keep the objects within reach as they land. The accuracy required to passively avoid collisions is considerably better, as one would expect. This is the reason I put this learning phase later than the reach-limited one.
3. It is much harder to achieve this "passive collision avoidance" with the fountain than with the cascade, and is essentially impossible for 8 or more juggled in a fountain. The reason is that in the cascade the balls travel a greater horizontal distance from throw to catch. There is an alternative even-number pattern using synchronous crossing throws (colloquially know as the "wimpy" pattern) which has an error tolerance equal to that of the cascade and may be a better choice for collision-limited jugglers.

The passive method just boils down to throwing accuracy. How to improve it? Some of the suggestions in the section above on reach-limited juggling apply. Here I will only add the point that it is important to not be too tired when attempting hard tricks, since tiredness makes throws less accurate. I mention it here because you are getting longer runs at this stage of the learning process and exhaustion becomes an important factor. I used to practice 7 balls for an hour with no breaks and basically try to ignore my tired arms. I would get stuck in a frustrating cycle -- practice hard to try to achieve a goal, get tired, get frustrated at the resulting sloppy pattern, practice even harder, and so on. It is important to take a minute or two following a long run to rest. Although it can seem like a waste of time to practice slowly, productivity is better in the long run. Don't try to force the issue when you feel tired.

The other option is to avoid collisions actively. You see an errant ball and adjust subsequent throws to avoid it. For example, in the fountain one always encounters this problem: one of the throws isn't angled out far enough and becomes a big target for subsequent throws from the same hand. To actively avoid these collisions, you can throw outside the plane of juggling (my preferred method) or try a column throw on the outside (more appropriate for clubs). Of course all this has to become instinctive, which is ok because heaven knows you have more than your fair share of fallow brain tissue. If you're willing to devote serious blocks of your free time to numbers juggling, it's virtually guaranteed.

Considering again the fountain problem, the mostly likely balls to collide are balls thrown consecutively from the same hand -- envision a pair of balls then, where the first ball was thrown incorrectly. Now one possible problem is that for large numbers, the first ball is not yet at its peak (where your attention is focused) when the second ball in the pair is thrown. How can you throw the second one to avoid the first when the first isn't even clearly in view? It may be that you don't need to see the first ball clearly to spot a problem with it: either peripheral vision is sufficient, or your sense of touch on the release gives you feedback (much as a typist can feel when missteaks have occurred, without looking). It's hard to second-guess how your body does these things. At any rate, one possible remedy suggested by Boppo in his Numbers Notes is to look lower than the apex to get sooner information about errors; the price you pay is that you get less feedback regarding throw heights.

You can also do pretty well if you just don't bother to avoid collisions. There's a lot of space in the air, and if you're juggling small beanbags then you can get some long runs just by being lucky. For example, it turns out that if you could throw 9 full-sized (6 cm) balls with good form (have more to say about what this means below) and so that they always landed within reach, then you could qualify roughly 25% of the time! This will be considered in more detail below, but the lesson is that unless you're doing as well or better than this, ignore collisions and focus on throwing evenly, within reach, and with good form to your pattern.

The Endurance-Limited Phase

This is the catch-all that will eventually cause everyone to drop, and refers to mental as well as physical endurance. Physically, it is humanly possible to do 5 clubs for at least 45 minutes and 7 balls for 4 minutes, once the previous learning phases have been worked through (some might question whether Anthony Gatto is in fact human). Breathe regularly, and try to relax all those tense muscles you don't really need. One should also not underestimate the difficulty of maintaining mental focus for a long time while juggling. I think this is what kills me on endurance runs with 5 balls.

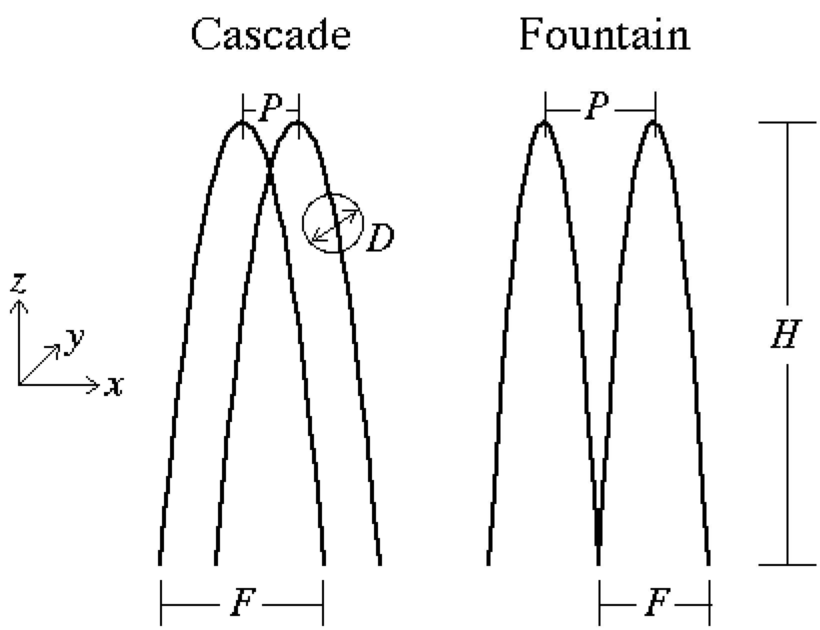
Quantifying the Demands of Numbers Juggling

The remainder of this paper will attempt to quantify the notion that "juggling requires good throws", and at the same time partially justify my list of the learning phases above. How good is good? If I get **X** throws with 7 balls, how many might I expect to get with 9, if I practice a lot? What should I focus on, to improve my juggling? These are the types of questions we're trying to understand.

Note: some of the sections below are a little technical, although the conclusions are not. Skim at will.

The Pattern

At this point let's introduce some notation for physically describing patterns. For convenience I will use the same notation as Jack Kalvan in Optimal Juggling. Refer to the following figure:



H is the height of the pattern, **F** is the horizontal distance between throw and catch, **P** is the distance between arcs, and **D** is the diameter of objects juggled. Also, define **N** to be the number of objects juggled. The final quantity of interest here is the dwell ratio **r**, which is the fraction of time that each hand spends filled by a ball. This ratio is typically between 0.60 and 0.65.

Throws and Throwing Errors

In toss juggling the balls undergo parabolic paths from throw to catch. This spatial path is completely determined by six quantities: the position **(x, y, z)** at the throwing point, and the ball velocity (v_x, v_y, v_z) at the throwing point. It is important to understand that for a given pattern there is an ideal value for each quantity, which the juggler is attempting to consistently obtain. The quantities **(x, y, z)** are relatively easy to achieve, as they are just the position of the hand at the throw point. The velocities (v_x, v_y, v_z), however, are determined by a fairly complex process in which hand speed, the amount of scoop on the ball carry, and the timing of the release all play important roles. I have found, by measuring jugglers on videotape, that inconsistency in the throw position is usually negligible compared with inconsistency in imparted velocity (throwing angle and height). An equivalent way to say this is that the variation in the catching position is always much greater than the variation in the throwing position.

If we number each throw in a juggling pattern with an index **i**, define the throwing velocity of the i^{th} throw to be (v_x^i, v_y^i, v_z^i). Of course this throw will not have a velocity precisely equal to the ideal one (v_x, v_y, v_z). Define a set of (dimensionless) throwing errors (e_x^i, e_y^i, e_z^i) for throw number **i** with:

$$(v_x^i, v_y^i, v_z^i) = (v_x, v_y, v_z) + (e_x^i, e_y^i, e_z^i) * v_z$$

Note that, for a nearly vertical throw (as in numbers juggling), e_x^i and e_y^i are the angular throwing errors in radians along the **x** and **y** directions, respectively. e_z^i is the fractional error in the upward velocity, and amounts to half the fractional error in the throw height **H**.

If we imagine tabulating these throwing errors for a particular juggler and pattern, they cluster about some average value in a bell-curve way, with some typical distance from the average given by the width of the "bell" (statistically, they are *normally distributed* or *Gaussian distributed* with a *standard deviation* corresponding to the width of the bell curve). Define:

$\mathbf{e_x}$ = standard deviation of the $\mathbf{e_x^i}$'s

$\mathbf{e_y}$ = standard deviation of the $\mathbf{e_y^i}$'s

$\mathbf{e_z}$ = standard deviation of the $\mathbf{e_z^i}$'s

$\langle \mathbf{e_x^i} \rangle$ = average of the $\mathbf{e_x^i}$'s

$\langle \mathbf{e_y^i} \rangle$ = average of the $\mathbf{e_y^i}$'s

$\langle \mathbf{e_z^i} \rangle$ = average of the $\mathbf{e_z^i}$'s

Hopefully, if a juggler has good form then the average values will be small relative to the widths (standard deviations) of the distributions. **This is in fact what we mean by "good form".** If this is not the case then the juggler's consistent (or systematic) throwing errors outweigh the random ones, and the juggler would be (at best) form-limited.

The parameters $\mathbf{e_x}$, $\mathbf{e_y}$, and $\mathbf{e_z}$ can be measured from videotape by looking at the departure of a throw from its ideal throw height and landing position. Newtonian mechanics gives the following:

$\mathbf{(delta\ H)^i = 2 * H * e_z^i}$ (error in throw height)

$\mathbf{(delta\ X)^i = 4 * H * e_x^i + F * e_z^i}$ (error in **x** coordinate of catch)

$\mathbf{(delta\ Y)^i = 4 * H * e_y^i}$ (error in **y** coordinate of catch)

In practice you can usually ignore the **F** term in the second equation. Measure the throw heights and landing positions relative to some convenient fixed position and then take the standard deviation of the numbers; solving the above equations will then yield the standard deviations $\mathbf{e_x}$, $\mathbf{e_y}$, and $\mathbf{e_z}$ (the choice of coordinate origin for the measurements drops out). This procedure is only effective for relatively stable patterns; if the pattern is drifting around it will overestimate the throwing errors if the averaging is done over long timescales. Other, more complicated, measurement schemes could be devised in that case.

I have found by making measurements on jugglers that these standard deviations are typically quite close to one another. For this reason I define the *throwing angular error* **E** as:

throwing angular error (radians) = $\mathbf{E = e_x = e_y = e_z}$.

This is a single number which quantifies one's throwing accuracy, in angular units. It may be somewhat confusing that we are expressing the error in the vertical velocity in angular units, since there seems to be no angles involved. In truth, the throwing errors $\mathbf{e_x}$, $\mathbf{e_y}$, and $\mathbf{e_z}$ are just dimensionless numbers which in the case of the first two can be conveniently interpreted as errors in the launch angle of the ball.

Because this is a nice memory mnemonic I use the term *angular error* for all of them.

As an example, measurements taken from a 1991 videotape of Anthony Gatto indicate that for both 7 and 9 balls, **E** is around 0.7° for him. If you're interested, use the above formulas to calculate what this means for his accuracy in throwing height and catching position; Optimal Juggling contains a convenient list of pattern heights and so on.

Aside Regarding *Optimal Juggling* and Error Surfaces

In Optimal Juggling, Jack Kalvan defines what he calls an *"error surface"* for the balls in their path (his parameter **D** is the diameter of this error surface, centered on a ball in flight). The size of this error surface is such that it will contain most, but not necessarily all, balls in flight. (That is, if you marked the position of each ball thrown by one of the hands at some particular time offset **t** after being thrown, you would eventually form a fuzzy region where the balls are clustered, and more accurate jugglers would have a smaller region. Kalvan assumes that this region is a sphere of some diameter **D**.)

From our present description it is clear that this error surface is the result of errors in throwing velocity, and we are led to the same concept as Kalvan with three differences:

1. The error surface is three-dimensional, whereas Kalvan assumes a two-dimensional surface (with balls that always remain in a perfect plane).
2. The error surface is not necessarily circular but in general ellipsoidal. However, as mentioned previously the standard deviations $\mathbf{e_x}$, $\mathbf{e_y}$, and $\mathbf{e_z}$ are typically quite close to one another, so a spherical error zone is a reasonable approximation.
3. The error surface is not constant in size, since errors in velocity give errors in position which grow linearly over time. Newtonian mechanics tells us that the error zone retains its shape throughout its flight, but will grow in proportion to the time **t** since the throw. Kalvan does not take this into account when deriving his optimal patterns (he assumes a constant error zone diameter **D**), although he does comment that the reverse cascade is harder than the cascade because errors have had longer to accumulate before the crossing point is reached -- a recognition of this fact that the error surface grows larger in flight. In practice, most collisions occur near the top of the pattern between adjacent balls with nearly equal error zone diameters, so his assumption is fairly reasonable for his purposes.

Avoiding Collisions and Maintaining Good Form

As discussed in the section on collision-limited juggling above, one technique for avoiding collisions is the passive one -- to throw so accurately that the objects never collide. In a JUGGLEN posting back in 1991 I discussed how accurate you have to throw to achieve a guaranteed collision-free pattern. The maximum allowed angular throwing error along the **x** axis is shown in the following table.

Number of Balls	Pattern Height (m)	Max. Allowed Error (degrees)
5	1.0	4.5
6	1.5	1.0

7	1.6	2.0
8	2.2	0.37
9	2.2	1.1
10	3.5	0.13
11	3.5	0.52
12	5.0	0.042

Here I have assumed that the juggler is using 6 cm diameter balls and is juggling with a "typical" armspan. This is again a *maximum* allowed error, as was discussed above in the section on reach-limited juggling. In other words, if you always throw within this angle of the ideal, you will never get collisions. For the more realistic case of Gaussian-distributed errors, your typical angular error **E** should be a factor of two or so smaller in order to ensure that collisions are infrequent. The formulas for calculating these allowed errors are:

$$E_{max} = \begin{cases} \frac{1}{4H}\left(\frac{2F}{N-1} - D\right) & \text{(fountain)} \\ \frac{1}{4H}\left(\frac{2(F+P)}{N+1} - D\right) & \text{(cascade)} \end{cases}$$

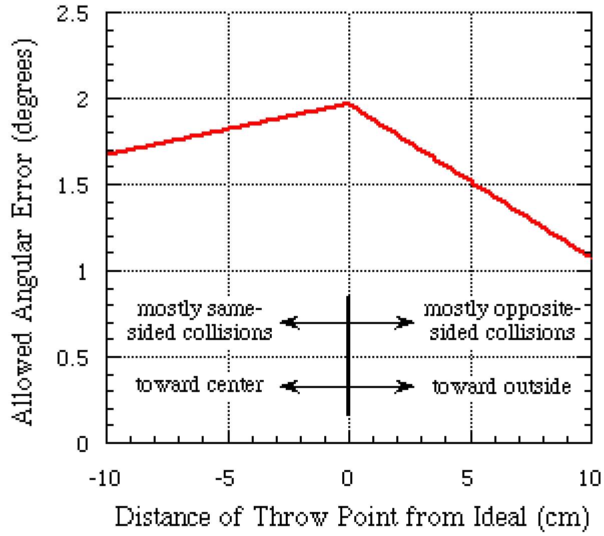
These formulas give results in radians (multiply by 57 to convert to degrees). Also, they make the approximation that there is no error in the throw heights of the balls.

The first point to note in the table above is that the even-numbered patterns are very unforgiving with regard to error tolerance. From the formula it is clear why:

1. The "baseline" **F** of each arc is smaller in the fountain, since you are throwing to the same side. In the cascade, you get the entire width of your body contributing to the arc baseline. A larger baseline is good because it provides more horizontal distance between balls in the arc when they peak.
2. The fountain is higher than it should be, because the balls are carried a greater distance than in the cascade and the juggler is forced to adopt a slower throwing rhythm. Most people, in fact, juggle 8 fountain at the same height as 9 cascade. In the formulas above **H** appears in the denominator -- higher patterns require more accuracy.

My conclusion is that if a juggler is going to consistently avoid collisions with a fountain of even 6 balls, it can only be done using an active type of method.

Some readers may have noticed something puzzling about the formula above for the cascade -- it only contains the total armspan **(F+P)** . Shouldn't the allowed error depend on where the throwing points are, as well as the total distance **(F+P)** between the catching points? In fact it does; this expression is for a pattern with optimal form. When the throwing points deviate from the ideal then the pattern tolerates less error. The following graph illustrates this for 7 balls, with the same parameters used in the table above:



The importance of good form is apparent from this graph -- a deviation of only five centimeters (a few inches) in the throwing points from their ideal positions makes a substantial difference. I have also pointed out on this graph how the types of collisions changes as you move the throwing position. If the throwing point is too far to the center, then most collisions will be between balls out of the same hand. Too far to the outside, and balls from opposite hands are the main problem. (This effect is noted by Jack Kalvan in his Optimal Juggling paper.) If you can tell which type of collision you're mostly getting when you juggle the cascade, you could tune your throwing point accordingly using this idea.

A Model Juggler, Simulated Within a Computer

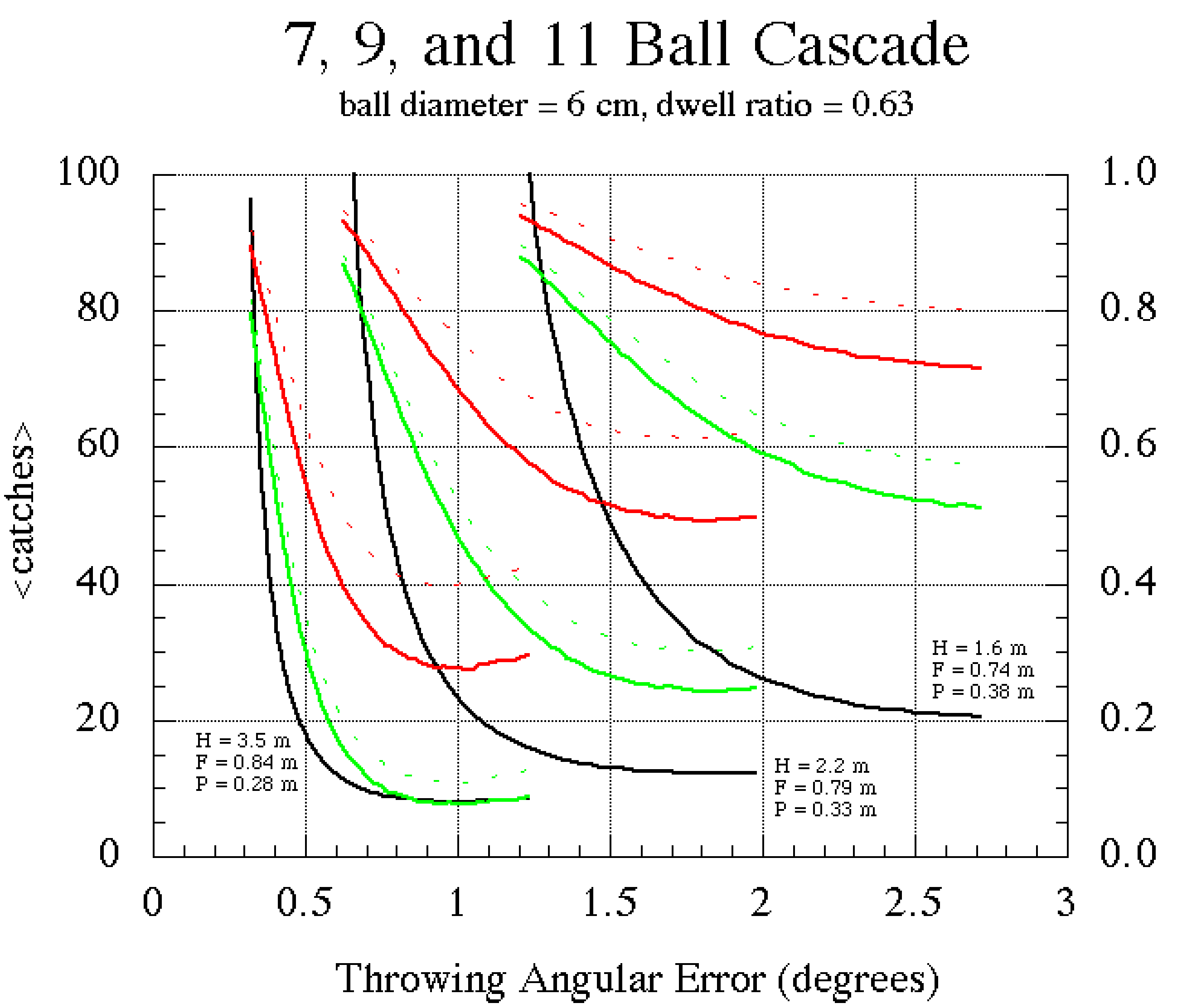
Now that we have a way to quantify (and actually measure) the throwing errors that a juggler makes, it would be interesting to see how these affect real performance: average run lengths, probability to qualify, and so on. In the tables throughout this paper I have listed the maximum allowed error that can be tolerated to avoid certain problems, such as a balls landing out of reach or collisions occurring. I said things like, "If you *always* throw with less error than this, you will never get collisions."

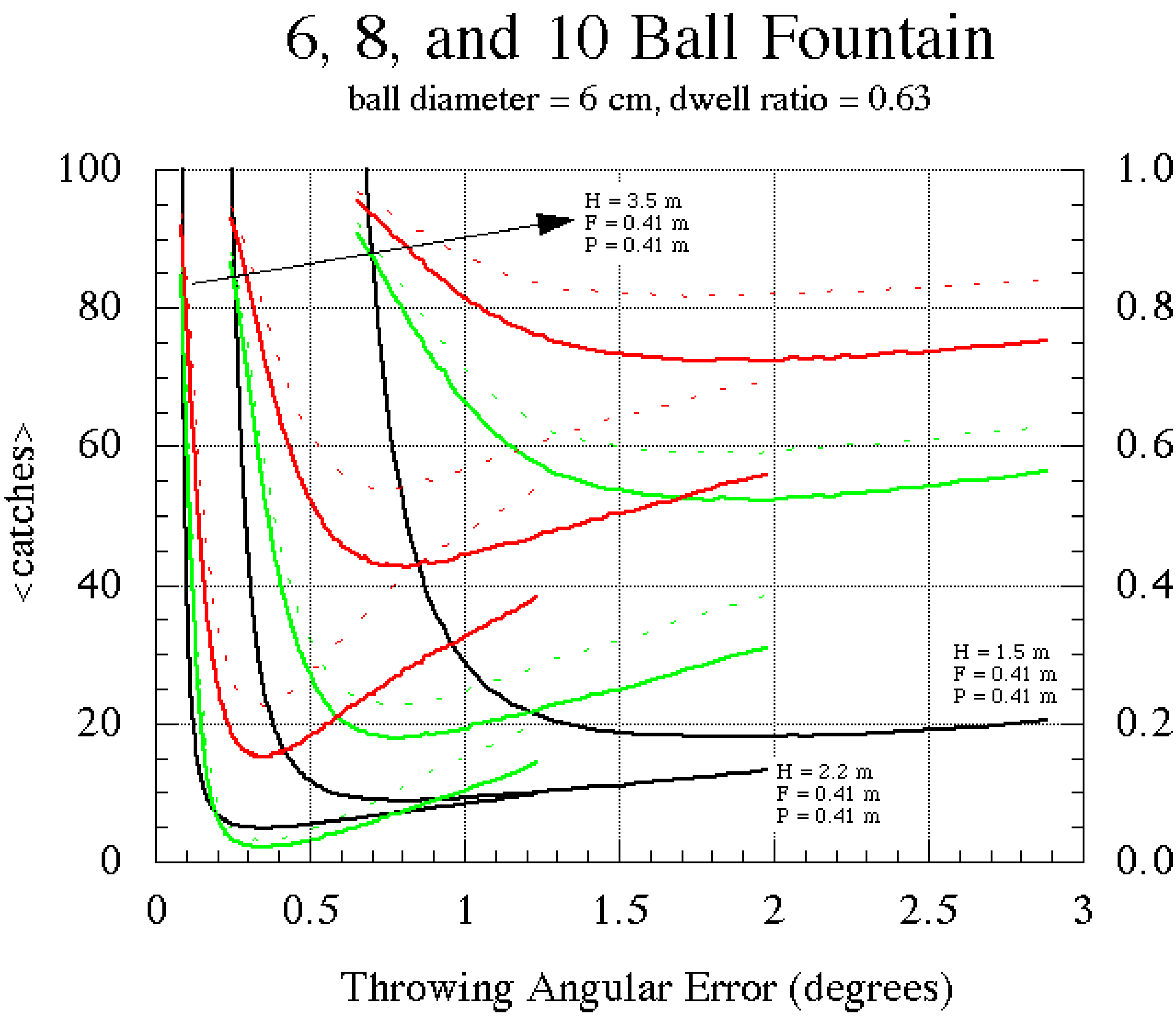
The problem with phrasing things this way is that no matter how good you are, you will never *always* throw within some accuracy limit. In practice, throwing errors follow the bell-curve distribution that was discussed above. Everybody makes big errors sometimes -- the question is, what's your *typical* throwing error, on any given throw?

I conducted a computer simulation of a numbers juggler, using our assumption of Gaussian-distributed throwing errors. (The C code is available in the References section below.) For a particular juggler accuracy **E**, this simulation consists of applying random Gaussian throwing errors to each throw in turn, and then following the balls in time to check for collisions. When a collision between balls occurs, it is assumed that neither object is caught, and we stop counting catches after the first missed catch (in accordance with standard practice). This is done for many runs to build up statistics for average number of catches, the probability of flashing (at least **N** catches), and the probability of qualifying (at least **2N** catches).

The results for both cascade and fountain are shown in the graphs below. For clarity I've put the cascade patterns on one plot and the fountains on another. The black curves indicate the average number of catches our simulated juggler gets, as a function of throwing error (7 balls is the rightmost black curve on the first graph, for example). The corresponding solid red and green lines show the fraction of the time the pattern was successfully flashed and qualified, respectively -- here it is assumed the juggler always continues throwing until a drop is made. The dashed curves indicate the probabilities when the juggler tries to collect after **N** (or **2N**) throws (the former is sometimes called a *straight flash* or *clean flash*). Error values past the rightmost extent of each line correspond to throwing errors so large that the average ball lands out of reach (more than 12 inches, or 30 cm, from the intended landing point). For smaller errors it is assumed that all balls which come down are caught and thrown again.

The values of **F** and **P** used in the cascade plot were determined by maximizing the average number of catches, subject to the constraint that the total pattern width **(F+P)** be kept fixed. **These values represent ideal pattern form.** As was discussed above, my results disagree slightly with those obtained by Jack Kalvan due to our differing treatments of throwing error.





Results of the Calculation

Each curve in the graphs above consists of four important regions. These correspond to the phases of learning a juggler experiences as she improves (her throwing error **E** decreases):

1. At the rightmost (highest **E**) limit of a pattern's curve, the juggler's throwing inaccuracy is large enough that many catches are difficult because the balls are landing far from where they should. When the throwing error is exactly at the rightmost endpoint on the curve, the error in the landing position is a foot (30 cm). Near this right endpoint, the juggler is reach-limited.
2. As the juggler improves and **E** gets smaller, she gets to a large relatively flat region. Here the balls are landing within reach, but random collisions cause drops. It may surprise people that average run length doesn't change much as throwing gets more accurate. For the fountain, a juggler that is not so accurate can even have the advantage -- a more accurate juggler confines the balls to a better plane and so doesn't get the advantage of random "collision-avoiding out of plane" throws discussed earlier. This is a collision-limited juggler without collision avoidance.
3. As **E** decreases still further, the average run length and probability to qualify begin to increase rapidly. Here the juggler is throwing well enough to get a passive sort of collision avoidance.

4. Eventually our hypothetical juggler, as she improves, will be capable of such long runs that she will get tired in the process. This is the endurance-limited phase.

From these plots one may also gain some insight about the relative difficulty of various numbers. For example, if your average 7 ball run is 50 catches then your throwing error **E** is roughly 1.5°. Using this knowledge, as well as the assumption that this throwing error is independent of number, you could conclude that you might achieve 13 or 14 catches on average with 9 balls, given sufficient practice (you have to work out the release, strength, timing, and form for this to really mean anything). To get so you can qualify a majority of the time with 9 (18 catches), your throws will have to be good to the point that you can do 7 balls essentially indefinitely.

Notice how difficult the fountain looks on these graphs. On the face of it, it looks as though 6 fountain were as hard as 9 cascade! Of course this is not the case. The point is that while the fountain is very collision-prone, you can to some extent avoid these collisions by detecting your own mistakes and throwing to compensate. This type of active technique is neglected here. So what do these graphs mean for the fountain? Probably not a lot, although they do give the "worst case performance" one might expect from a collision-limited juggler -- for example, if you are getting a successful flash on less than 40% of your attempts with 8 balls (of the size used in the simulation, of course), then you are probably a reach-limited 8 ball juggler. The fountain results also underscore the crucial role that active collision avoidance plays in sustaining these patterns.

Personally, these graphs give me hope because they rise rapidly below a threshold throwing error. There is a lot of work getting to that threshold, mostly unrewarded (the curves are pretty flat to the right), but there is some truth to the idea that things "click" once you get good enough. This breakthrough may not happen so quickly as it did with 3 balls ("Oh, I get it!"), but it can occur nonetheless.

Assumptions and Limitations of the Model

The virtue of the present model is that the ability of the juggler has been distilled down to a single measurable number, **E**. Four important simplifying assumptions have been made in the process:

1. The pattern maintains good form at all times, meaning that throws are always made from the ideal position and that the average errors $\langle e_x^i \rangle$, $\langle e_y^i \rangle$, and $\langle e_z^i \rangle$ are all small relative to **E** (as many throws are too high as are too low, for example).
2. There is no active type of collision avoidance.
3. All landing balls are caught.
4. The errors made from one throw to the next are uncorrelated -- a mistake in one throw doesn't increase the likelihood of mistakes later on. In other words, errors don't compound themselves.

Each of these assumptions is clearly an oversimplification of real juggling. They have been made because to fully deal with these issues would require at least a biomechanical model of the catch/scoop/throw process -- no simple task. How does the body actively detect and avoid collisions? What types of mistakes cause a pattern to degrade, and which can be easily corrected? None of these important questions is addressed here, and their answers may in fact vary from juggler to juggler.

You might think about how each of these assumptions may have biased my results. For example, assumption 2 causes the simulation to overestimate the difficulty of the fountain pattern, and will be violated by any decent 6-ball (fountain) juggler. Assumption 3 will obviously break down near the rightmost extent of each curve (where the average ball lands a foot, or 30 cm, from where it should); we must keep in mind that the curves above are certainly too high in this region. Do you find assumption 4 valid for your own juggling -- are most of your errors "sudden", or do you feel the pattern decay gradually over many throws as errors accumulate? I would guess that assumption 4 would be bad near the right end of each curve, since in my experience mistakes are harder to correct quickly when you're racing all over the place to make catches.

It would be an interesting line of research to develop a more realistic juggling model than we've done here, one which would relax some of our assumptions. For example, regarding assumption 3 we might conjecture that each juggler has a particular catching radius or maximum handspeed which could be used to determine when a particular catch can't be made. We could develop techniques to measure these limitations for actual jugglers, and conduct a more accurate simulation (of course the juggler is now quantified by more than just a single number **E**, so the results might be harder to visualize). This exercise could yield more detailed training advice than what's provided here.

Training Summary

The seven phases of learning are summarized here.

Remember: "Why do I usually drop?"

Release-Limited	Goal:	Learn to hold the objects in your hands and release them consistently
	Try:	• simple, consistent grip • higher, slower throws at beginning
Strength-Limited	Goal:	Learn to launch the objects high enough for the pattern, while maintaining consistency on the release
	Try:	• weight training • juggling with wrist weights or heavy balls
Timing-Limited	Goal:	Learn to throw to an even height so that the balls land in a regular rhythm
	Try:	• "drop test" to listen for rhythm • ignore balls in the air
Reach-Limited	Goal:	Learn to throw so that the objects consistently land close enough to the desired catching position
	Try:	• don't let mistakes change your throws ("let the pattern come to you") • don't throw to avoid objects in the air • concentrate on peaks or crossing point and aim there

Form-Limited	Goal:	Learn the proper form and "feel" of the pattern
	Try:	• videotape yourself and review it carefully • watch good jugglers on tape • concentrate on maintaining the correct throwing point and target • don't let badly-placed objects in the air break your form • visualize during off-practice time • end on a good run
Collision-Limited	Goal:	Learn to throw so that the objects don't collide in the air
	Try:	• rest between runs • when you break form to avoid a collision, pull the pattern back into correct form quickly • look lower in pattern, away from peaks?
Endurance-Limited:	Goal:	Juggle for as long as possible
	Try:	• breathe regularly • relax unneeded muscles

I would again reiterate that these aren't really such neatly-defined boxes. For example, you might want to look at your form earlier than indicated above, especially since it can be hard to train yourself out of bad form once it's ingrained. (Nevertheless from a physical standpoint if you're reach-limited then most of the times you drop will be from the pattern falling apart because you had to scramble around too much to make catches (with subsequently bad throws), not from bad form per se.) Also, you might find videotape or weight training helpful at any point along the learning curve.

The most clearcut piece of information I've gotten from all this analysis is that, unless you're doing really well (and if you are would you call it "numbers"?), try to ignore the objects in the air. Don't throw to avoid collisions, but rather focus on maintaining the correct throwing points and throwing targets. When do you start thinking about moving beyond this? The following table indicates roughly how good you should be (with 6 cm diameter balls) before you worry at all about avoiding collisions:

Number of Balls	Average Catches	% Runs Qualify
6	20	55%
7	20	50%
8	15	30%
9	13	25%
10	10	15%
11	10	10%

Conclusions

Numbers juggling is a difficult, complicated thing for the body to learn. Especially near the beginning of the learning process, it is easy to become overwhelmed by the fact that everything seems to be going wrong. The way to overcome this is to simplify matters where you can -- figure out what is most holding you back, and concentrate on improving that.

As an example from my own juggling, I have been spending time lately working on the 8 ball fountain, a pattern at which I'm reach-limited. It is pointless at this stage to make an effort to avoid collisions -- it just diverts my attention from what is important, throwing consistently so I can catch the balls. I visualize the two peaking points in space and try to ignore the balls in the air as I throw. I find that this simple mental picture helps me quite a bit sometimes when the whole thing is falling apart. Now with 6 balls this is not my mental image at all -- there I am thinking about the ideal peaking points with the corner of my mind, but am also observing the pattern and "bending it around" to correct for my mistakes, trying to pull it into good form but yielding a bit when my own mistakes force it.

Numbers juggling demands self-reflection, both of one's physical movement and one's mental state. This is especially true for those of us who are self-taught. We are forced to be analytical, perhaps overly so at times. As a physicist I have found it natural to break up the learning process into increasingly-difficult stages, as dictated by the various physical failure modes. Within this picture each stage has a natural training regimen, a most productive path toward progress. I would be interested to see how others might subdivide this problem -- would a psychologist, for example, identify different mental hurdles to be overcome and make training suggestions from that basis? Others might focus on the juggler's form, feeling that to be of paramount importance, and would train accordingly. I suppose ultimately my only justification in thinking that my method of breaking down the problem has some element of truth is that I can calculate the difficulty in numerical terms and identify that certain things about juggling are physically more demanding than others. And who can argue with numbers, especially when juggled so well? I can't believe I just wrote that.

References

1. Numbers Notes, by Bruce Tiemann (aka Boppo).
2. Optimal Juggling, by Jack Kalvan.
3. A JUGGLEN posting of mine from 1991.
4. errors.c, the C program used in the simulation.